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Preservice Mathematics Teachers' Ability to Perform the Mathematizing Process: The Cylinder Packing Problem

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The present study aims to examine the preservice middle school mathematics teachers' ability to perform the process of mathematizing and to identify their competencies within this context. For this purpose, the study was conducted with 43 preservice teachers attending a state university. The research method used is descriptive research. As the data collection tool, a real-life problem called "cylinder packing problem" was presented to preservice teachers, with a view to finding the optimal and lowest-cost packing scheme for a hazelnut grower. The solutions presented by the preservice teachers were subjected to content analysis as well as descriptive analysis. The preservice teachers' competencies were examined to see their level of effectiveness in the use of notations that are indicators of mathematizing, in achieving mathematical results in the process of problem-solving. The research findings indicate that the preservice teachers generally lacked sufficient comprehension of the problem in the real-life contexts and could not perform the requirements of important skills such as using notations, mathematical models and generalization method which are important in making abstract inferences in the mathematizing process. In light of these findings, further studies to contribute to preservice teachers' mathematical sophistication levels are recommended in order to improve their competencies in mathematizing.

Introduction

Supporting the comprehension of mathematical concepts within the applicable real-life contexts, unlike the traditional approaches to mathematics education, mathematizing (Rosales, 2015) is basically the process of organizing the information at hand with various strategies, and thereby achieving mathematical generalizations (Freudenthal, 1973). Freudenthal (2002b), who defines the task of a mathematician as problem-solving, problem posing, and mathematizing events, states that students can also undertake this task, and that the main thing in mathematics teaching is to teach mathematization, not mathematics (Freudenthal, 1968). Writing "There is no mathematics without mathematizing", Freudenthal (1973, p. 134) underlines the importance of mathematizing, and inspired several studies on mathematics education.

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While mathematics is recognized as a deductive science in general, many mathematical results are reached by proving the generalizations achieved through induction (Polya, 1973; Yıldırım, 2008). Based on noticing patterns through a limited number of observations and defined as a certain type of inductive reasoning, generalization (Mason, Burton, & Stacey, 2010) is addressed at the middle school level, while the deductive approach is the subject of high school mathematics classes. In generalization, one starts with small examples essentially based on observation, conclusion, formulation, etc., following the principle of deduction, and acts with the approach of heuristic thinking (Polya, 1973; Yıldırım, 2008). Heuristic approach aims to find appropriate methods and rules in the process of exploration and invention (Polya, 1973), is an integral part of problem-solving in terms of improving and enhancing the reasoning skills (Mousoulides & Sriraman, 2014). According to this approach, it is essential to apply intuitive means to achieve the best result, while knowledge or learning attained thus is easier to retain (Gordon, 1962).

The Link between Real Life Problem Solving and the Mathematizing Process

Along with offering the means for rather conventional application, problem-solving also helps with mathematical thinking and enables acquiring mathematical information, which plays a crucial role in mathematizing (Archambeault, 1993; Freudenthal, 2002b; Polya, 1973; Schoenfeld, 1992; Tall, 2006). Problems are known as real-life problems in the literature function as a bridge between real life and mathematics and make mathematics a rather meaningful occupation (Boaler, 1993; Clarke & Roche, 2009). According to the approach of Realistic Mathematics Education pioneered by Freudenthal, problem-solving aims to acquire mathematical knowledge regarding the real-life situation (Gravemeijer, 2008; Gravemeijer & Terwel, 2000; Van den Heuvel-Panhuizen & Drijvers, 2014). In this mathematizing process, formal mathematical knowledge is acquired through references to informal knowledge covering elements of real problem situations (Freudenthal, 1968, 2002b). Mathematizing is a horizontally and vertically successive process: in horizontal mathematizing elements of the real world move into the world of symbols, and in vertical mathematizing abstraction takes place (Freudenthal, 2002b). In other words, whereas horizontal mathematizing leads to results based on various problem-solving strategies and the concrete problem case, in vertical mathematizing, these results are organized and generalized within an abstract construct, therefore leading to formal mathematical results (Gravemeijer, 2008; Rasmussen, Zandieh, King, & Teppo, 2005). Mathematizing is needed when solving real-life problems, and this process is usually described as modeling and completed with the help of various notations (Yerushalmy, 1997). According to Kant and Sarikaya (2021), notation is a key component of mathematizing and can be used to understand a situation through representations such as numbers, sketches or symbols.

Real-life modeling activities that support the mathematizing process help students to perceive mathematics as a value by enabling them to comprehend the concrete relationship between mathematics and the effort put, and also improve their critical thinking, problem-solving, cooperation, and communication skills (Suh, Matson, & Seshaiyer, 2017). These activities also offer an effective learning environment (Lady, Utomu, & Lovi, 2018). However, the ability to address real-life situations in a mathematical context requires substantial preparation on part of the teachers, for finding or formulating appropriate problems, as well as applying them in the classroom (Clarke & Roche, 2009; Freudenthal, 2002a; Gravemeijer, 2008; Suh, Matson, & Seshaiyer, 2017). The observation that teachers can apply mathematical rules does not necessarily indicate that they know how to come up with those rules (Freudenthal, 1968, 1973). The fact that the teachers can actualize the process of mathematizing, defined as the process of obtaining knowledge and rules, can be considered an indicator of their ability to apply an alternative approach to the formation of knowledge.

Purpose

In Türkiye, the middle school mathematics curriculum is shaped around crucial concepts such as mathematical literacy and mathematical modeling, which allow one to work on real-life applications of mathematics (Ministry of National Education [MoNE], 2018). These concepts, in turn, are introduced on the basis of problem-solving and association skills which constitute integral parts of mathematics curricula. The middle school mathematics curriculum (MoNE, 2018) emphasizes at several points that the association between life and mathematics can contribute to meaningful mathematics teaching. Along the same lines, the Content-Specific Competencies of Mathematics Teacher text specifies teacher competencies regarding the problem-solving and association skills. Some are noted in the quotation below (MoNE, 2008, p. 144):

> Knows the importance of problem-solving skill's contribution to mathematics learning. […] Enables students in terms of questioning the problem-solving process and confirming the results they came up with. […] Knows the significance of an awareness of mathematics' connections within and with other disciplines as well as with daily life, in terms of contributing to learning mathematics and reflects this awareness on her practices.

Managing the mathematizing process based on concrete events in line with the recommendations of decision makers in education, requires significant skills and preparations on part of the teachers, as indicated in previous research. Nevertheless, there is only a very limited number of studies performed with teachers and preservice teachers, on how mathematics education in Türkiye is actually being carried out (Tabak, 2019). In this context, the present study aimed to examine the preservice middle school mathematics teachers' process of acquiring mathematical information within the context of a real-life problem and to identify their competencies within this context. To that end, the preservice teachers were assessed to see their level of effectiveness in the use of notations that are indicators of mathematizing (Kant, & Sarıkaya, 2021), in achieving mathematical results in the process of solving a given real-life problem.

Method

As the study aimed to explore an existing situation in the existing state of affairs, the descriptive research method was utilized (Knupfer, & McLellan, 1996). Since the aim of the research was to reveal an existing situation in its reality, descriptive research method was used in accordance with this purpose. In descriptive research, events are defined, data are tabulated and explained with descriptions (Glass & Hopkins, 1984). The solutions provided by the preservice teachers for the given real-life problem constituted the research data. As the data were to be qualitatively analyzed, the study can be considered an example of qualitative research.

Participants

The participants of this study were 43 preservice teachers (14 men and 29 women) attending the third grade of middle school mathematics teaching program at a state university in Türkiye.

Pre-service teachers were selected with the convenient sampling method, one of the purposive sampling methods. A prerequisite for inclusion in the sample is that the participant has taken a course on problem-solving. First, 53 preservice teachers' comprehension of the problem who took lessons and volunteered for the research were evaluated, and it was found that ten of them could not comprehend what was expected of them in the problem. Hence, these preservice teachers were excluded from data analysis, and the solutions of 43 preservice teachers were examined in detail. In the research, the preservice teachers were coded as T1, T2, T3... to ensure confidentiality.

Data Collection

The data of this study were collected through a real-life problem. The collection tool was created by the researcher through the adaptation of a problem in the literature, to a real-life problem. How to roll an A4 paper to get the largest cylinder in terms of volume is a classic problem in the literature (Basden, et al., n.d.; Joye-Bortolotti & Vilmart, 2013). Although cylinders formed by connecting the edges of an A4 paper horizontally or vertically as illustrated in Figure 1 have the same lateral surface areas, their volumes are different.

Figure 1. In what way should we roll an A4 paper so as to get the cylinder with the greatest volume? (Joye-Bortolotti & Vilmart, 2013, p. 1).

This problem was then envisaged as a real-life problem in the form of a packing problem. For years, packing problems have been drawing the interest and curiosity of researchers (Fasano, 2014). They are often formulated to require the placement of two- or three-dimensional objects in a given area or volume in the most optimal way possible (Birgin, Martínez, & Ronconi, 2005). In real life, especially in industry, these problems aim to find the most economic configuration of products in terms of area or volume as well as packing cost, in various operations including packaging, storage or transportation (Castillo, Kampas, & Pintér, 2008). In this sense, the problem was posed by the researcher as the data collection tool is about looking for the most suitable option of packing the product with the lowest packing cost, for a hazelnut grower. The cylinder packing problem used as the data collection tool is as follows:

> A hazelnut grower wants to offer her product in a cylindrical container.
> A) Rectangular cardboards of a standard size will be used for each product, as the lateral surface of the cylinder. How should the grower, who wants to minimize the packing cost, use the cardboard? Mathematically prove the validity of your proposal.
> B) Base and cover of the container will be cut out of an aluminum sheet. Is your proposal still valid, taking these costs into account, as well? Please provide proof.

The hazelnut grower wants to launch her product in a cylindrical container, made of cardboard in terms of its lateral surface, and of aluminum in terms of its base. No dimensions are provided in the problem, for the cylindrical container. The first step (A) of the two-step problem is about the lateral surface, and the second step (B) is about the bases (see Appendices for a detailed description of the mathematizing processes regarding the solving of the problem, and possible mathematical results).

In the first step (A) which tries to find the condition required to minimize the product's packing cost, one needs to arrange the use of the packing material so that the packaging produced provides the greatest volume for the product. To that end, since the lateral surface of the cylinder is made of rectangular cardboard, one should try to find which edges of the cardboard to connect to meet the required condition. In addition to the written instructions, the participants were verbally told that rectangular cardboard cannot be cut into smaller pieces and shall thus form the lateral surface in one piece. On the other hand, since the participants were asked to confirm the result mathematically, they are also expected to actualize the mathematizing process. In this context, two starting points that should be identified when looking for the solution in step A of the problem, and that is decisive in problem's solution are as follows:

> A1. It is necessary to arrange the packing material so that the packages produced correspond to the greatest-volume product.
> A2. The cardboard can be used in two different ways to form the lateral surface of a cylinder.

Identifying the starting points A1 and A2 through the association of the problem data is an indicator of understanding the context of the problem. In this problem that can be solved by means of algebraic expressions as no measurements are provided, one can commence the work on the solution by identifying the variables, through actualization of vertical mathematization. The solution process leads to the first mathematical result as follows:

> Result 1: The volume of a cylinder formed by bringing together the short edges of a rectangle is greater than the volume of a cylinder formed by connecting the long ones.

In the second step (B), it is stated, as a second condition, that bases of the cylinder packing are to be cut out of an aluminum sheet. Here, the result reached in step A is expected to be in light of packing cost, from two potential aspects: Firstly, the bases of the packages (cylinders) are cut out of aluminum sheet (B1), and secondly, the packing cost is to be evaluated in terms of volume (B2). The process leads one to the second and the third (based on the problem context) mathematical result(s) as follows:

> Result 2: One can place identical circles on a rectangular area, with the minimal amount of wasted space in between, is to place them in a hexagonal arrangement.
> Result 3: When the base material of the cylinders costs more than the lateral surface material, cylinder packages formed by connecting the long edges of the cardboard cost less.

For data collection, the preservice teachers were asked to individually and in writing solve the problem, which was presented to them in writing, in the classroom, out of course. They were allowed to use as much time as they needed to do so. In general, they used for two hours. In the problem-solving process, it was ensured that the researcher made no intervention so that the situation could be addressed in its whole reality.

Data Analysis

The study focused on preservice middle school mathematics teachers' process of acquiring mathematical knowledge within the context of a real-life problem. Accordingly, the data analysis was performed in two steps. In the first step, the notations used by the preservice teachers in their solutions for the problem were subjected to content analysis. In the second step, these notations were analyzed descriptively in terms of their accuracy. Content analysis allows for classification and interpretation of data, which cannot otherwise be explored in plain reading, through an analysis based on a methodological, systematic and quantitative approach (Cohen, Manion, & Morrison, 2007; Mialaret, 2004). Thereafter, descriptive analysis was used to obtain and summarize problem-related data from pieces of text, only to be organized under similar categories to come up with qualitative inferences (Cohen, Manion, & Morrison, 2007).

The extent to which the preservice teachers took crucial information into account the information, and the process through which they reached mathematical results were analyzed. This examination found five distinct notations used for reaching the mathematical results: Experimental, visual, verbal, numerical and algebraic. These notations were at times utilized individually, and in groups in others. A three-point scale (0: incorrect, 1: incomplete, 2: correct) was used in the scoring of solutions. Scoring results were arranged in frequencies and percentages, followed by a discussion of the results. Notations and scoring are explained below.

- Experimental notation: It is about utilizing the suggestion of comparing the volumes of different cylinders, by using sand or similar materials.
- Visual notation: This notation refers to using drawings in setting up the cylinder. Such drawings, however, do not play an important role other than illustrating and supporting the source of data used in the solution. The presentation of the cylinder in from two distinct angles was accepted as correct while its presentation from just one angle was considered to be incomplete.
- Verbal notation: It refers to providing a verbal explanation in writing. Since verbal explanation needs to be based on a mathematical inquiry in the process of mathematizing, the success of this process depends on the correctness of numerical or algebraic notations employed therein.
- Numerical notation: This notation is about representing edge lengths of a rectangle, with numbers. For example, edge lengths of rectangle can be represented as "$a = 12$ units" and "$b = 6$ units". If the generalization was based on a single example, it was considered incomplete; if a few different values were used, it was considered correct.
- Algebraic notation: This notation is all about using algebraic calculations. Two different categories of algebraic notation were identified: variables and algebraic expressions. Variables were used to represent edge lengths of the rectangle with letters, while algebraic expressions were used to represent them with alphanumerical characters. Variables representing the edge lengths of the rectangle included a and b or x and y whereas $2a$ and a or h and $2\pi r$ were among the algebraic expressions. When the use of algebraic expressions was prone to incorrect generalizations, the case was considered to be incomplete. For example, when the edge lengths of rectangle are represented as a and $2a$ respectively, a two-fold relation would also appear between the volumes based on these measurements. Even though such a finding does not affect the general outcome, such a result would still be erroneous. On the other hand, since no faulty relation occurred when lengths were taken as h and $2\pi r$, these representations were considered correct.

Some examples of data analysis are presented below. Figure 2 presents the solution of T20 who utilized just the visual, verbal, and experimental notations in Problem A.

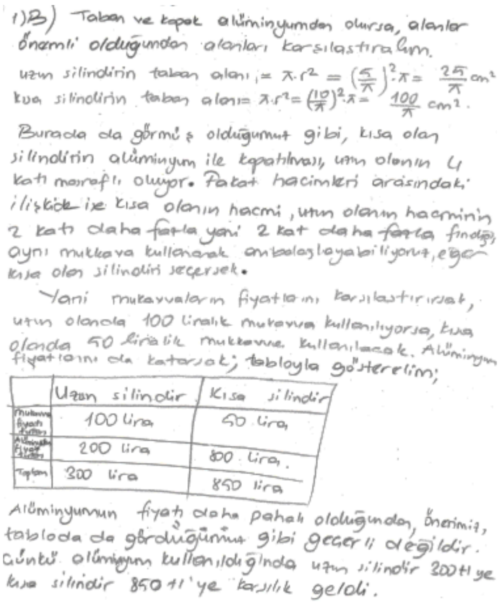
Translation of the text in the figure:

There are two options, as seen in the figure.

A) To confirm this proposition through experimentation, when we fill cylinders I and II with sand and water and pour out the sand out and weigh them, we see that cylinder in figure I takes more sand. Because the cylinder in figure I has a greater volume than the one in shape II. Base area of the cylinder in figure I is larger than the base area of the one in figure II.

Figure 2. Example of the correct use of visual, verbal, and experimental notations in Problem A's solution (T20).

Although notations used by T20 were correct, it was accepted that the mathematizing process was not implemented as there were no numerical or algebraic notations. An example considered incomplete is presented in Figure 3.


1)B) Taban ve kapak alüminyumdan olursa, alanlar önemli olduğundan alanları karşılaştıralım.
uzun silindirin taban alanı = $\pi \cdot r^2 = \left(\frac{5}{\pi}\right)^2 \cdot \pi = \frac{25}{\pi}$ cm²
kısa silindirin taban alanı = $\pi \cdot r^2 = \left(\frac{10}{\pi}\right)^2 \cdot \pi = \frac{100}{\pi}$ cm².
Burada da görmüş olduğumuz gibi, kısa olan silindirin alüminyum ile kapatılması, uzun olanın 4 katı masraflı oluyor. Fakat hacimleri arasındaki ilişki ise kısa olanın hacmi, uzun olanın hacminin 2 katı daha fazla yani 2 kat daha fazla fındığı, aynı mukavva kullanarak ambalajlayabiliyoruz, eğer kısa olan silindiri seçersek.
Yani mukavvaların fiyatlarını karşılaştırırsak, uzun olanda 100 liralık mukavva kullanılıyorsa, kısa olanda 50 liralık mukavva kullanılacak. Alüminyum fiyatlarını da katarsak; tabloyla gösterelim;

	Uzun silindir	Kısa silindir
Mukavva fiyatı tutarı	100 lira	50 lira
Alüminyum fiyatı tutarı	200 lira	800 lira.
Toplam	300 lira	850 lira

Alüminyumun fiyatı daha pahalı olduğundan, önerimiz, tabloda da gördüğümüz gibi geçerli değildir. Çünkü alüminyum kullanıldığında uzun silindir 300 TL'ye kısa silindir 850 TL'ye karşılık geldi.

Translation of the text in the figure:
B) If the base and the cap are to be made of aluminum, we should compare the areas, as they are important.
Base area of the long/short cylinder:
$\pi \cdot r^2 = \left(\frac{5}{\pi}\right)^2 \cdot \pi = \frac{25}{\pi}\,\text{cm}^2 \qquad \pi \cdot r^2 = \left(\frac{10}{\pi}\right)^2 \cdot \pi = \frac{100}{\pi}\,\text{cm}^2$
As we see here, capping the short cylinder with aluminum costs 4 times more than the comparable cost item for the long cylinder. But regarding the relation between volumes, the volume of the short one is 2 times greater than that of the long one; that is, we can pack 2 times more hazelnuts using the same cardboard if we choose the short cylinder.
To compare the cost of the cardboards: assuming you pay 100 liras for the longer cardboard, and 50 liras for the shorter one... If we add the cost of aluminum, the following table appears:

	Long cylinder	Short cylinder
Cardboard cost	100 liras	50 liras
Aluminum cost	200 liras	800 liras
Total	300 liras	850 liras

As aluminum costs more, our proposition is not valid as seen in the table. For, when using aluminum, the long cylinder costs 300 liras while the short one costs 850 liras.

Figure 3. Example of the incomplete use of numerical notation in Problem B's solution (T18).

T18 assumed the edge lengths of the cardboard to be *10 cm* and *20 cm* respectively: therefore, accepting the base radiuses as *5 cm* and *10 cm*. Although the participant reached the correct result in the end, the solution was considered to be incomplete due to the use of a single numerical example.

Certain precautions were taken to achieve validity and reliability in various steps of the study. In the light of research ethics concerns, voluntary participation and the privacy of the preservice teachers were deemed crucial. During data collection, no time limit was set for the preservice teachers to solve the problem; and they were provided verbal clarification about the problem, when necessary. The suitability of the cylinder packing problem used as data collection tool for the research problem, and the kind of method used in data collection and analysis were explained in detail, with definitions, coding, and examples. The findings were presented with clear and complete coding and citations to enable the review thereof by researchers. Moreover, two experts specialized in mathematics teaching were consulted; and required corrections were affected in the light of recommendations about accuracy and clarity of the study.

Findings

Findings were grouped under the categories "the process of mathematizing" relating to Problem A and Problem B.

Findings Regarding the Process of Mathematizing in Solving Problem A

To go through the relevant mathematical processes and achieve Result 1 (The volume of a cylinder formed by bringing together the short edges of a rectangle is greater than the volume of a cylinder formed by connecting the long ones) which is the mathematical result expected for Problem A, one needs to understand that problem is about volume (A1: It is necessary to arrange the packing material so that the package produced corresponds to the greatest-volume of product) and that cylinders can be obtained by using a rectangle in two distinct ways (A2: The cardboard can be used in two different ways to form the lateral surface of the cylinder). Relevant findings are given in Figure 4 and Figure 5.

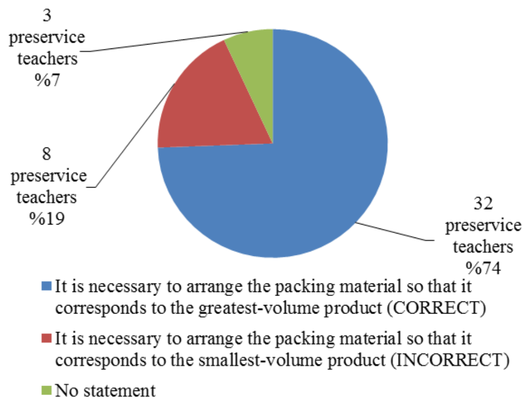


Figure 4. Findings on the statement A1.

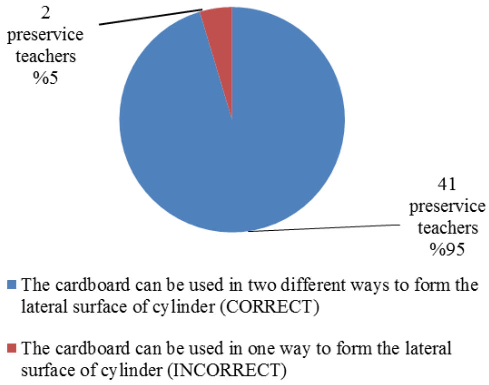


Figure 5. Findings on the statement A2.

Examining the data regarding the extent of the preservice teachers' consideration of A1 and A2 which played a key role in the problem solution led to the finding that

- 10 preservice teachers who incorrectly handled or did not declare only information A1, naturally achieved the incorrect result,
- Two preservice teachers who handled A1 and A2 (T37), or just A2 (T34) incorrectly reached the incorrect result as they did not compare the volumes of the cylinders,
- 32 preservice teachers processed both information correctly. Two of them (T17 and T39) made calculation errors (i.e., forgot squaring the pi; and assumed base radiuses of the cylinders to be equal) which could affect the result when calculating the volume and thus reached incorrect results.

Notations used in the solution of Problem A

The distribution of notations utilized by the preservice teachers in solving Problem A is presented in Figure 6.

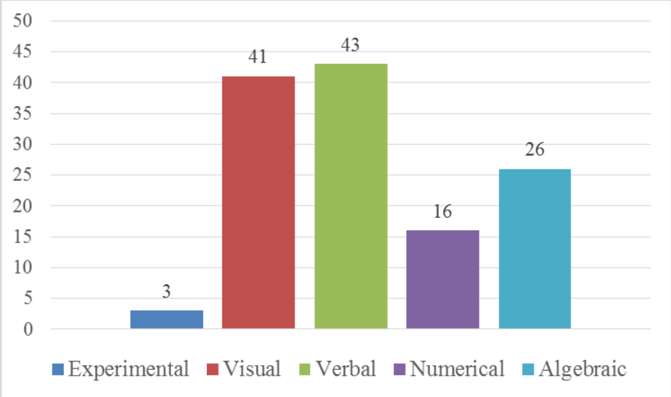


Figure 6. Distribution of notations utilized in Problem A's solution.

It was observed that all preservice teachers with the exception of T7 and T19 often used verbal and visual notations in conjunction. In addition to these notations, with the exception of T30 and T33, they opted for either numerical or algebraic notations. In other words, among the preservice teachers, 14 preferred numerical notations while 24 chose to use algebraic notations. T8 used numerical and experimental notations whereas T7 and T20 only used the experimental notation in combination with the visual and verbal notations. Figure 7 presents the distribution of notations used for solving Problem A, by correctness.

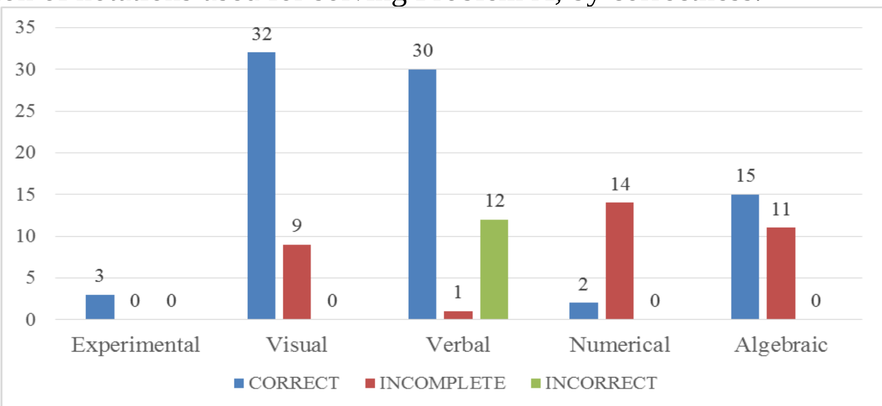


Figure 7. Distribution of notations utilized in Problem A's solution, broken down by correctness.

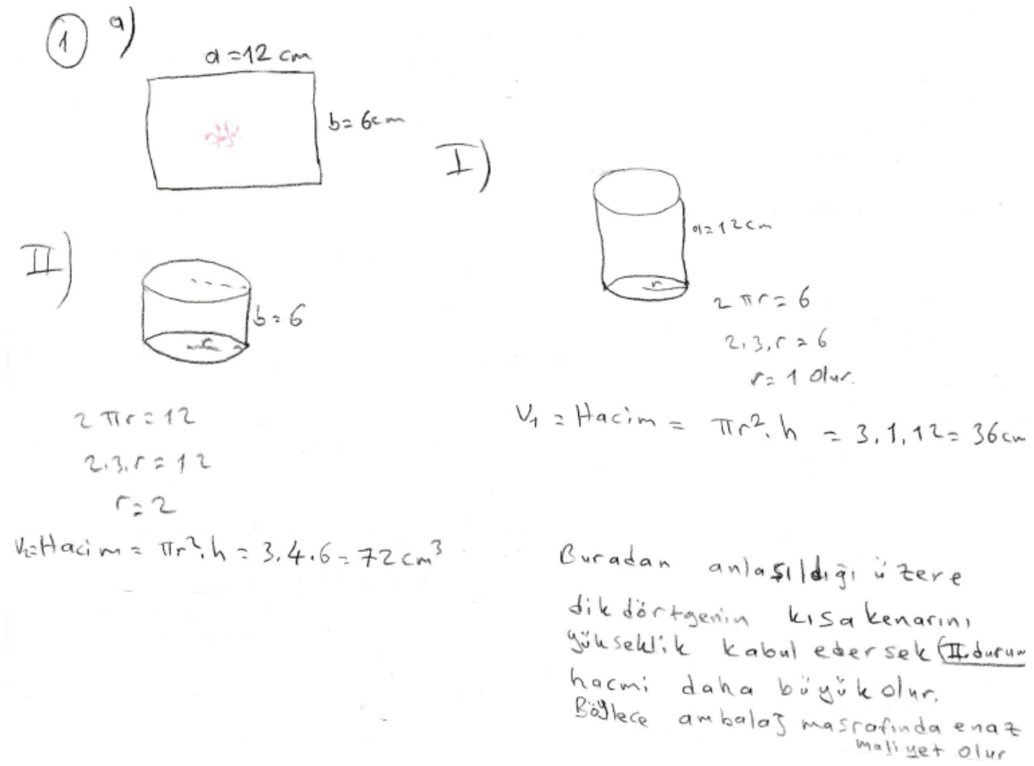
Rates of correctness for the notations are as follows, respectively: Experimental (100%), Visual (78%), Verbal (70%), Algebraic (58%), Numerical (13%). Nine of the preservice teachers used the visual notation incompletely by drawing only one of the cylinder images formed by connecting the long or the short edges of the rectangle. However, as these preservice teachers verbally stated that cylinder could be formed in two different ways also by connecting different edges, it was concluded that a total of 41 preservice teachers had the information A2. While the preservice teachers' use of the experimental notation was always correct, the highest number of errors was observed with verbal notations. 30 verbal notations considered correct indicates that about three-fourths of the preservice teachers came up with a correct result. However, variation from this rate is still possible, given that the success in the mathematizing process depends on the correctness of numerical or algebraic notations. 17 of the numerical and algebraic notations are correct. Therefore, about one-third of the preservice teachers carried out a correct mathematizing process. According to Figure 7, the preservice teachers who used numerical or algebraic notations usually preferred only one of them. 15 out of 26 preservice teachers who used the algebraic notation and 2 out of 16 preservice teachers who used the numerical notation carried out the operations correctly. Almost all of the numerical notations were found to be incomplete. In the case of algebraic notations, however, the number of correct operations was closer to that of incorrect ones. These findings are analyzed with reference to the values of numerical notations used by 16 preservice teachers as presented in Table 1.

Table 1. Numerical Notations used for Cardboard Dimensions and Scoring.

T	1	2	3	8	13	17	18	26	27	28	30	33	36	38	39	42
a	60	12	20	12	10	8	20	8	12	6	G	G	12	4	8	10
b	30	4	10	6	6	5	10	4	6	3	G	G	6	2	1	8
S	1	1	1	1	1	1	1	1	1	1	2	2	1	1	1	1

Note. *T* = preservice teacher; *a*, *b* = cardboard dimensions; *G* = use of different values for generalization; *S* = scoring; *1* = incomplete; *2* = correct.

In terms of numerical values, more often than not, a two-fold relation was preferred in the context of the edge lengths of the rectangle. The two-fold relation between the values was also observed in terms of volume values, supporting the generalization. For example, the numerical values used by T36 were $a = 12\ cm, b = 6\ cm,$ expressing a two-fold relationship (Figure 8).



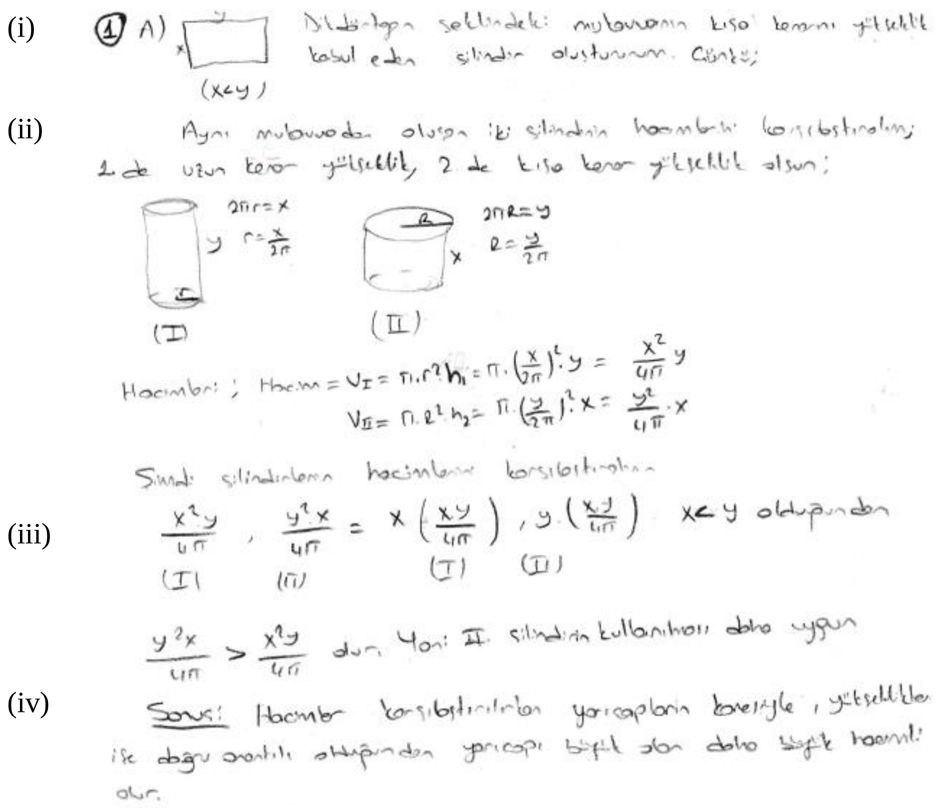
Translation of the text in the figure:
As understood from here, if we accept short edge of rectangle as height (situation II), its volume is greater. Therefore, packing cost becomes lower.

Figure 8. Example of the incomplete use of numerical notation in Problem A's solution (T36).

However, the answer provided by this preservice teacher is not considered a complete one, as examples with different values were not provided.

With the exception of two preservice teachers, the generalizations were always made with just one numerical value, and therefore, are considered incomplete. After having calculated the volume based on just one numerical value, T30 and T33, who used the numerical notation correctly, suggested that this operation could be deductively repeated with different values.

Among algebraic notations, 18 used variables while 8 used algebraic expressions. Variables were used to represent edge lengths of the rectangle with letters, while algebraic expression was used to represent them with alphanumerical characters. An exemplary solution utilizing the variables is given in Figure 9.



Translation of the text in the figure:
(i) I form a cylinder accepting short edge of rectangular cardboard as height because...
(ii) Let us compare volumes of two cylinders made of the same cardboard; let the long edge be height in (I) and the short edge be height in (II).
(iii) Since x < y, it is $\frac{y^2x}{4\pi} = \frac{x^2y}{4\pi}$. That is, it is more preferable to use cylinder II.
(iv) Result: Comparing the volumes, because square of radiuses is in direct proportion to heights, the one with greater radius has a greater volume.

Figure 9. Example of the correct use of visual, verbal and algebraic (variable) notations in Problem A's solution (T29).

In Problem A's solution, T29 expressed cardboard edges with variables x and y and also used visual and verbal notations correctly (Figure 9).

Table 2 shows the algebraic notations used by the preservice teachers for cardboard dimensions, along with the scoring.

Table 2. Algebraic Notations used for Cardboard Dimensions and Scoring.

T	9	11	21	23	24	30	41	43
a	*H*	*h*	$4\pi r$	*2a*	$2\pi r$	*h*	$2\pi r_1$	*2a*
b	$2\pi r$	$2\pi r$	$2\pi r$	a	$2\pi r'$	$2\pi r$	$2\pi r_2$	a
S	2	2	1	1	2	2	2	1

Note. *T* = preservice teacher; *a, b* = cardboard dimensions; *S* = scoring; *1* = incomplete; *2* = correct.

Rather than assigning the variables a and b to represent the edge lengths of the rectangle, T24 and T41 assigned the algebraic expressions of $2\pi r$ and $2\pi r'$ instead. Since notations allowed for obtaining base radiuses of the cylinders, therefore facilitating the calculation, this approach can be considered a smart and valid one. The notations presented by T21, T23 and T43, on the other hand, were considered incomplete, as designating the notations as a and $2a$ interdependently might cause misleading generalizations such defining them as multiples of each other (Table 2).

In the case of Problem A, the solution was considered incomplete when there were important errors or incompleteness, other than calculation errors in algebraic notations. For example, after assuming the edge lengths to be $a > b$, T40 assumed the base radiuses to be $a/2$ and $b/2$, and proceeded with the algebraic operation. Although the calculation was based on incorrect values, it still did not change the result. However, the mathematizing process is assumed to be correct.

Findings Regarding the Process of Mathematizing in Solving Problem B

None of the preservice teachers attempted to reach 2 (One can place identical circles on a rectangular area, with the minimal amount of wasted space in between, is to place them in a hexagonal arrangement) which is one of the mathematical results expected for Problem B. Findings about the anticipated Result 3 are also presented below.

Notations utilized in the solution of Problem B

Figure 10 presents the distribution of notations utilized by the preservice teachers to get Result 3, one of the mathematical results expected for Problem B.

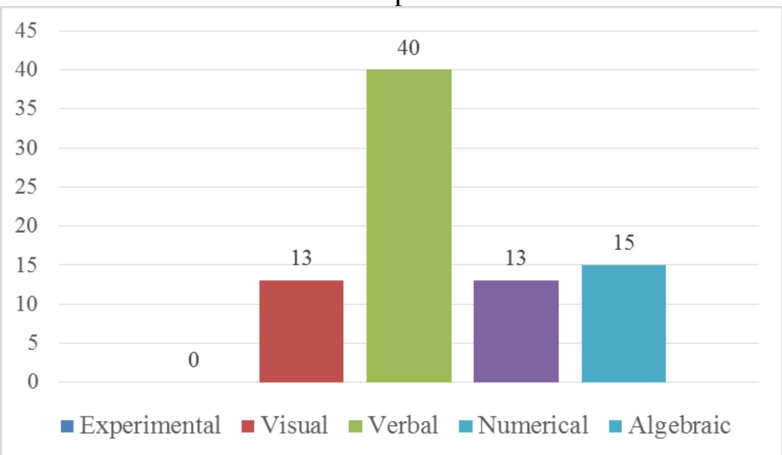


Figure 10. Distribution of notations utilized in the solution of Problem B.

While none of the preservice teachers utilized the experimental notation in the solution of Problem B, almost all of them (except T1, T40, and T42) used the verbal notation. One-third of the preservice teachers were found to utilize the visual notation, and all but T11 preferred either the numerical or algebraic the notation. In other words, among the preservice teachers who took part in the study, 12 preferred numerical and 14 preferred algebraic notations, whereas 13 produced their solutions without using numerical or algebraic notations. About one-third of the preservice teachers did not employ the mathematizing process.

Figure 11 presents the distribution of notations utilized for Problem B’s solution, by correctness.

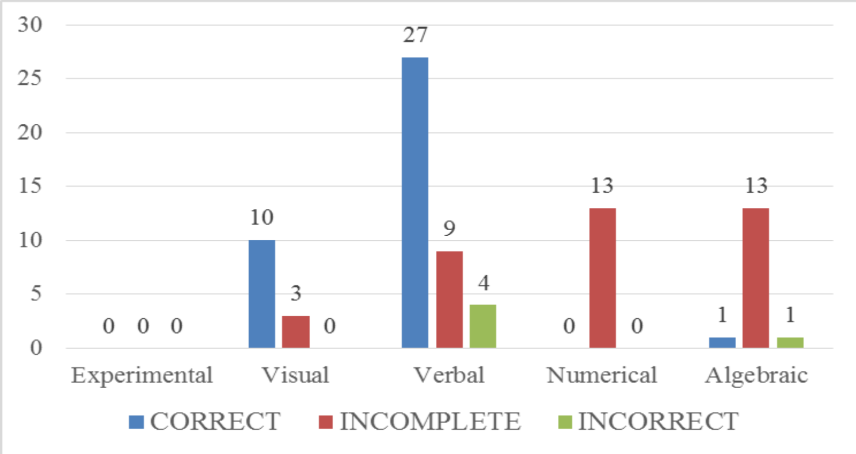
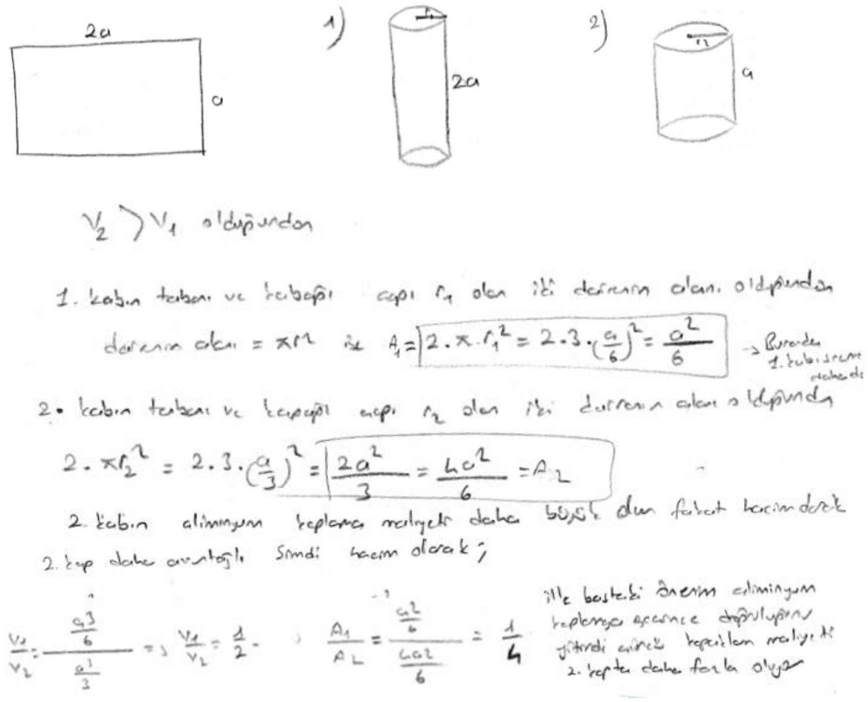


Figure 11. Distribution of notations utilized in the solution of Problem B, broken down by correctness.

According to the data presented in Figure 11, although roughly three-fifths (27/43) of the preservice teachers came up with the correct result using the verbal notation, they were unable to support this result with numerical or algebraic notations. Only one preservice teacher (T23) used a correct algebraic notation (Figure 12).

(i)
(ii)
(iii)
(iv)
(v)



Translation of the text in the figure:

(i) As the area of container 1's base and cap is the area of two circles with a diameter of r1, if area of circle is πr2,

(ii) It is better to choose container 1.

(iii) As the area of container 2's base and cap is the area of two circles with a diameter of r2,

(iv) Aluminum covering cost of container 2 is more, but container 2 is more advantageous for the volume. As for the volume,

(v) My first proposition lost its validity when we proceeded with aluminum covering because cost of caps goes up in container 2.

Figure 12. Example of the correct use of algebraic notation in Problem B's solution (T23)

Thus, it is understood that only T23 correctly performed the mathematizing process among all preservice teachers who took part in the study. However, it should be noted that, when beginning his/her solution, T23 assumed the edge lengths of the rectangle to be a and $2a$. In verbal notations considered incomplete, the preservice teachers usually used informal logic during their efforts to achieve the correct result. That is to say, the preservice teachers emphasized the price difference between the materials (cardboard vs. aluminum) and preferred the model of cylinder with smaller radius which permitted using less aluminum. In numerical notations considered incomplete, the participants were observed to try to make an inference through a single numerical example. In most cases, they disregarded volume as a variable to take into account. An example is presented in the data analysis section (Figure 2).

It was observed that the preservice teachers resorted to comparing the base areas without engaging in any comparison between the volumes, of different cylinder models.

Discussion

The study explored whether the preservice middle school mathematics teachers could successfully perform the mathematizing process, which is defined as the process whereby applicable knowledge and rules can be put into practice concerning a real-life problem. The study was carried out with 43 preservice teachers enrolled in a state university and utilized as its data collection tool a problem looking for the most optimal packing option with the lowest packing cost. The solution of the problem called the "cylinder packing problem" requires actualizing mathematizing with reference to three mathematical results.

The preservice teachers were observed to use five different notations (experimental, visual, verbal, numerical, and algebraic) in the problem-solving process. These notations are examined as an indicator of mathematizing for achieving the mathematical results. To be able to achieve Result 1 which is the mathematical result expected for Problem A, one needs to know that the problem is about volume, and that the cylinder can be obtained from a rectangle in two different ways. The findings initially revealed that 13 preservice teachers who utilized only one of the two statements correctly, or who made calculation errors, on the other hand, came up with incorrect results in the mathematizing process. It is understood from these findings that, grasping the problem and the ability to associate variables in the problem played a key role in problem solving, and that 30 preservice teachers, about three-fourths, had these skills.

To achieve Result 1 expected for the solution of Problem A, the preservice teachers usually used verbal and visual notations together and preferred either numerical or algebraic notations for use in combination with these notations. Whereas all preservice teachers used verbal notation, the least frequently used notation was the experimental notation. Regarding the correctness of these notations, one can confidently state that all instances of experimental notation use were correct, while the rates of correctness for visual, verbal, algebraic, and numerical notations are 78%, 70%, 58%, and 13% respectively. Those who used the algebraic notation were found to be more successful than those who used the numerical notation. Previous research emphasized that algebraic thinking and abstraction play an unavoidable role in making mathematical generalizations (Davydov, 1990; Dumitrascu, 2017). Since abstraction requires mental reorganization, it is more complex than generalization which involves accustomed processes (Tall, 1991). In a study investigating the mathematical thinking levels of preservice teachers, Alkan and Bukova Güzel (2005) found that preservice teachers were incompetent at using the generalization method, and often resorted to performing operations, which led to negative outcomes in abstraction.

Among the preservice teachers who used the numerical notation, all but two applied generalization based on a single numerical value. Producing one example is considered adequate only when used for falsification purposes, according to the counter-example method (Hammack, 2018). As often at least four or five examples are called for generalizations (Polya, 1973), one can forcefully argue that the preservice teachers were unable to use the numerical notation effectively. As one of the ways of accessing information in school mathematics, generalization is considered the lifeline of mathematics (Mason, Burton, & Stacey, 2010). It is emphasized that mathematical thinking will not occur as long as teachers are not aware of generalization, regarded as the core of mathematics, and do not give the students a chance to generalize (Mason, 1996). It was observed that the analysis of the fold relationship between lengths of the edges of the rectangle was often based on numerical values. Choosing the most suitable examples to comprehend and solve a problem occurs during the stage of specializing, which is among the basic processes of mathematical thinking, and it is expected to be followed by generalization (Mason, Burton and Stacey, 2010). Certain examples in the study might have been preferred by the preservice teachers to facilitate calculation and to assist generalization. However, such generalizations were mostly limited to the fold relation, indicating that successful applications of the generalizations were rare, if not non-existent.

In generalization, calculation with numerical data should be replaced by abstraction; only then mathematical deduction can be accepted as information. Most of the preservice teachers who used algebraic notation – the main indicator of abstraction– preferred variables such as *(a, b)*, while others used algebraic expressions such as *(h, 2πr), (2πr, 2πr'), (2a, a)* to represent edge lengths of the rectangle. The fold relationship between the edge lengths, which led to a limited set of generalizations, was utilized in the choice of algebraic expressions, just like the case with the numerical notations, and that notations such as *(2πr, 2πr')* were preferred to obtain more practical figures for the base radiuses of the cylinders, so as to facilitate calculation.

In the mathematizing process, visual and verbal notations are usually the tools to support and explain the solution and are not sufficient on their own. Experimental notation is utilized less frequently and, again, is incapable of reflecting the mathematizing process on its own. Given that a successful mathematizing process depends on the correctness of the numerical or algebraic notations, one can say that about one-third of the preservice teachers actualized a correct mathematizing process.

None of the preservice teachers attempted to achieve Result 2, one of mathematical results expected for Problem B. It is possible that the statement provided as Problem B, "Base and cover of the container will be cut out of an aluminum plate. When taking these costs into account, is your suggestion still valid? Provide confirmation" did not lead preservice teachers to thinking that multiple circles could be cut out of a plate. Furthermore, while there was a clear question about how to use the cardboard so as to minimize the packing cost in Problem A, the lack of such a question about the use of the aluminum plate might have caused this outcome. The preservice teachers neglected the cost issues in the problem and looked for a limited solution. This finding indicates that the preservice teachers only focused on the clearly stated questions and did not apply the step of posing a subproblem in the problem-solving process. According to Polya (1973), when solving problems involving practices in different domains of mathematics, previously acquired information in the relevant domain is required, yet, of the need for such information is more complex and ambiguous compared to those of mathematical problems.

The mathematical result expected to be reached in accordance with the problem context is a statement to the effect that the cost of cylinder packing formed by connecting long edges of the cardboard is lower when base material of the cylinder costs more than its lateral surface material (Result 3). Accordingly, in contrast to the result achieved in Problem A, the second problem is about achieving the minimum cost for unit of volume. It was observed that the preservice teachers did not use the experimental notation in Problem B's solution. This might have been the case because experimental notation does not provide any solution to the problem. Visual notation which is a supportive tool in comprehending the solution was, again, not a frequent occurrence in this problem's solution. Since the relevant visual notation was used in Problem A's solution in general, one might think that it was not needed much in Problem B's solution. Almost all preservice teachers used the verbal notation while about one-third of them preferred, in addition to the verbal notation, either numerical or algebraic notations. The preservice teachers who produced solutions without using either numerical or algebraic notations are not deemed to have actualized the mathematizing process correctly. Although about three-fifths of the preservice teachers achieved the correct result, the mathematizing process was regarded unsuccessful as they did not support this result with numerical or algebraic notations. For example, the preservice teachers usually referred to the price difference between the materials (cardboard/aluminum) and stated that a cylinder packing with a smaller radius and containing less aluminum should be used. While this result is correct, it is not possible to speak of a mathematizing process because it does not involve a mathematical explanation.

As it was found that, in general, the preservice teachers made efforts to make inferences through a single numerical example, one can argue, in parallel to previous findings, that they have significant shortcomings about generalization. Only one preservice teacher used verbal and algebraic notations correctly and employed a successful mathematizing process for the third mathematical result.

Conclusion and Suggestions

The research findings showed that the preservice teachers could not perform the requirements of important skills such as using notations, mathematical models and generalization method which are important in making abstract inferences in the mathematizing process and supporting the problem solution with mathematical models or explanations. Results similar to the findings of the present study were common in the literature. In two studies investigating the preservice teachers' mathematical literacy skills, it was observed that preservice teachers had difficulty associating variables in a problem, translating the problem situation to mathematical language, and interpreting it within the daily-life context, and failed to express the solution algebraically while using verbal representations (Kabael and Ata Baran, 2019; Kabael and Barak, 2016). In a comparative study, Uygur-Kabael (2017) observed that Turkish preservice teachers, when compared against their American peers, performed poorly in mathematizing processes such as creating quantities in real-life problems, association, and using suitable mathematical representations.

Seaman and Szydlik (2007) found that preservice teachers had inadequate levels of mathematical sophistication –a term referring to the command of mathematical descriptions used by the mathematics community, in problem-solving and instruction. According to the researchers, the approach used by the preservice teachers in problem-solving and instruction is closely related to their mathematical sophistication levels, mathematical experiences, and whether they can use the mathematical language which shapes their perspectives of mathematics. To boot, any improvements in regarding this level plays an important role in understanding how to teach mathematics. In parallel with current approaches in mathematics teaching, it is crucial to ensure that preservice and in-service mathematics teachers have the required competencies in taking real-life applications of mathematics to the school environment. Design of the environments so as to enable re-invention of mathematics, and improvement of conceptual understanding of how and why mathematics is developed requires diligence and attention (Lai, Kinnear and Fung, 2019). Although preservice teachers receive education shaped by the principles of realistic mathematics education, they remain incapable of putting it into practice in classroom and need assistance (Bozkurt, Kozaklı Ülger and Altun, 2019). Studies aimed at developing practical course materials and teacher and teacher candidates' skills such as problem- solving, and modeling can be recommended to contribute to teacher education. Research to identify and enhance mathematical sophistication levels of in-service and preservice mathematics teachers would certainly contribute to the literature.

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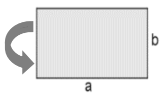
Appendices: A Priori Analysis of the Cylinder Packing Problem

Mathematizing Processes and Mathematical Results Regarding Problem A

Taking the volumes to be achieved based on a and b, once dimensions of the cardboard which will form the lateral surface of packing are chosen as a being the long edge and b being the short edge ($a > b$), and it is taken into account that the lateral surface of cylinder can be formed using the cardboard in two distinct ways:

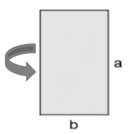
If we represent the volume to be obtained when short edges of the cardboard are connected, with $\mathcal{V}_1$,



Since $\mathcal{V}_1 = \pi {r_1}^2 h_1$, we get

$$\left.\begin{aligned} h_1 &= b \\ a = 2\pi r_1 \Rightarrow r_1 &= \frac{a}{2\pi} \end{aligned}\right\} \mathcal{V}_1 = \pi {r_1}^2 h_1 \Rightarrow \mathcal{V}_1 = \pi \left(\frac{a}{2\pi}\right)^2 \cdot b = \frac{a^2 b}{4\pi}$$

If we represent the volume to be obtained when long edges of the cardboard are connected, with $\mathcal{V}_2$,



Since $\mathcal{V}_2 = \pi {r_2}^2 h_2$, we get

$$\left.\begin{aligned} h_2 &= a \\ b = 2\pi r_2 \Rightarrow r_2 &= \frac{b}{2\pi} \end{aligned}\right\} \mathcal{V}_2 = \pi {r_2}^2 h_2 \Rightarrow \mathcal{V}_2 = \pi \left(\frac{b}{2\pi}\right)^2 \cdot a = \frac{b^2 a}{4\pi}$$

Since $a > b$, simplifying the relations obtained to compare $\mathcal{V}_1$ and $\mathcal{V}_2$,

$\mathcal{V}_1$	$\mathcal{V}_2$	
$\frac{a^2 b}{4\pi}$	$\frac{b^2 a}{4\pi}$	Multiplying the two expressions by 4π
$a^2 b$	$b^2 a$	Dividing the two expressions by ab
a	b	we get $\mathcal{V}_1 > \mathcal{V}_2$, where $a > b$,

which indicates that the internal volume of the cylinder packing formed by connecting short edges of the cardboard is greater. Thus, the first mathematical result is reached:

> The volume of a cylinder formed by bringing together the short edges of a rectangle is greater than the volume of a cylinder formed by connecting the long ones.

Mathematizing Processes and Mathematical Results Regarding Problem B

Cutting the bases of packing (cylinder) out of aluminum plate (B1)

The purpose of the activities known in mathematics as tessellation is to place objects or shapes onto a plane without any overlaps and gaps. When the shape to be placed onto the plane is a circle, one needs to investigate how the plane can be covered so as to leave the fewest gaps. A plane can be properly covered with isodiametric circles in two different ways. The first is the quadratic tessellation and the second is the hexagonal tessellation. Circle density on the plane is higher with the hexagonal tessellation (Weisstein, n.d.)[1]. Intuitively and perceptively, it is obvious that there would be less material waste with the hexagonal tessellation, and this implication can be confirmed algebraically. A solution approach is given in Figure 1.

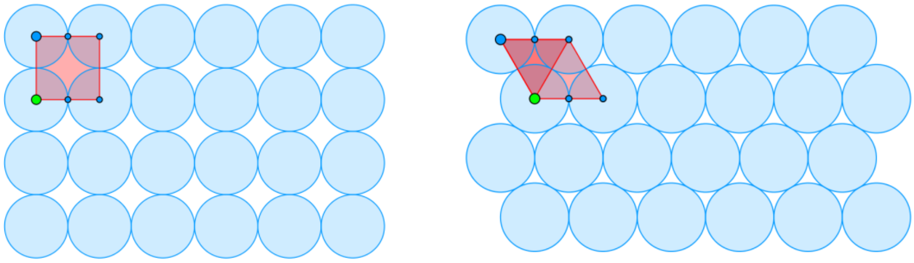


Figure 1. Comparison of the gaps between circles in the quadratic and hexagonal packing models of 24 isodiametric circles.

When the area of four quarter sectors is subtracted from the area of the square which is the combination of centers of four circles with a radius of r unit (u), the gap between the four circles is found to be $r^2(4-\pi)\,u^2$ in the quadratic model. In the hexagonal model, the gap between the four circles is found to be $r^2\left(2\sqrt{3}-\pi\right)\,u^2$ by subtracting the area of a circle from the area of a rhombus with edge length of *2r u* and height $r\sqrt{3}\,u$ (the height can be obtained from the equilateral triangle relation: $h = \frac{2r\sqrt{3}}{2} = r\sqrt{3}\,u$).

In the context of comparing the gaps left in both models, a $4 > 2\sqrt{3}$, one can argue that a hexagonal model with a gap of $r^2\left(2\sqrt{3}-\pi\right)\,u^2$ is the model that should be preferred to minimize the material cost. Thus the second mathematical result is reached:

> One can place identical circles on a rectangular area, with the minimal amount of wasted space in between, is to place them in a hexagonal arrangement.

Furthermore, when achieving this result, strategies suitable for the middle school level can be developed based on area relations of known geometric shapes such as triangles, quadrangles and circles.

Evaluating the packing cost in the context of volume (B2)

It is known from step A of the problem that internal volume of the cylinder packing formed by connecting short edges of the cardboard is greater; therefore, base area of the cylinder packing is greater than the base area of the packing formed by connecting long edges of the cardboard. In this case, the fact that more of the aluminum material would be used and aluminum material is more expensive than cardboard material necessitates the comparison of larger volume vs. minimum cost. Intuitively, by the condition of minimizing the cost allocated for packing material, it seems probable that smaller bases would be preferred over larger bases. Considering this case mathematically, circle area per volume (amount of aluminum

[1] Weisstein, E. W. (n.d.). Circle packing. The MathWorld Wolfram. Retrieved from http://mathworld.wolfram.com/CirclePacking.html

plate required) needs to be calculated. Using the previous formulas (where a: long edge of cardboard; b: short edge of cardboard; $\mathcal{V}_1$: volume created by connecting the short edges of cardboard; $\mathcal{V}_2$: volume created by connecting the long edges of cardboard), if each volume of $\mathcal{V}_1$ and $\mathcal{V}_2$ is to be compared to two base areas respectively, we get

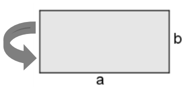
$$\left.\begin{array}{ll} For\ \mathcal{V}_1, & \dfrac{2\cdot\pi\left(\dfrac{a}{2\pi}\right)^2}{\pi\left(\dfrac{a}{2\pi}\right)^2\cdot b} = \dfrac{\dfrac{a^2}{2\pi}}{\dfrac{a^2}{4\pi}\cdot b} = \dfrac{2}{b} \\ & \\ For\ \mathcal{V}_2, & \dfrac{2\cdot\pi\left(\dfrac{b}{2\pi}\right)^2}{\pi\left(\dfrac{b}{2\pi}\right)^2\cdot a} = \dfrac{\dfrac{b^2}{2\pi}}{\dfrac{b^2}{4\pi}\cdot a} = \dfrac{2}{a} \end{array}\right\}\ we\ get\ \frac{2}{b} > \frac{2}{a}, \qquad where\ a > b$$

Consequently, it is seen that the circle area per volume is larger for $\mathcal{V}_1$ and the suggestion we have made in step A of the problem loses its validity. Thus, the third mathematical result based on the problem context is reached:

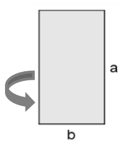
> When the base material of the cylinders costs more than the lateral surface material, cylinder packages formed by connecting the long edges of the cardboard cost less.

Moreover, one should remember that choice of larger circle would mean more waste of aluminum plate, which might be considered as a further increase in the cost of a large circle.

The lack of any measurements in the problem statement, and the use of materials of various value levels for packaging are the factors that make the solution different. As a different solution in such cases, it can be recommended to test the intuitive implication with easily computable values. For instance, if we take dimensions of the cardboard as *a = 20 u* and *b = 10 u* and calculate the volumes, we get



$$\left.\begin{array}{r} h_1 = 10 \\ 20 = 2\pi r_1 \Rightarrow r_1 = \dfrac{10}{2\pi} \end{array}\right\}\ \mathcal{V}_1 = \pi {r_1}^2 h_1 = \pi\left(\frac{20}{2\pi}\right)^2\cdot 10 = \frac{1000}{\pi}\ u^3$$



$$\left.\begin{array}{r} h_2 = 20 \\ 10 = 2\pi r_2 \Rightarrow r_2 = \dfrac{20}{2\pi} \end{array}\right\}\ \mathcal{V}_2 = \pi {r_2}^2 h_2 = \pi\left(\frac{10}{2\pi}\right)^2\cdot 20 = \frac{500}{\pi}\ u^3$$

In this numerical example, when the height is set to be long edge of the cardboard, it is seen that the volume of the cylinder is half the volume of a comparable cylinder with height is set to the short edge of the cardboard. If one seeks to calculate the amount of packaging required to launch the same amount of product, the following conclusions are reached:

> For $\mathcal{V}_1$: 1 cardboard + 2 aluminum bases of large circle
> For $\mathcal{V}_2$: 2 cardboards + 4 aluminum bases of small circle

Even if the cardboard is used two times more for a cylinder with the volume of $\mathcal{V}_2$, what is to be considered here is the amount of aluminum plate which costs higher than cardboard.

Calculating total area of individual aluminum bases, one would get

$$Area\ of\ large\ circle\ = \pi {r_1}^2 = \pi \left(\frac{20}{2\pi}\right)^2 = \frac{100}{\pi}\ u^2$$

$$Area\ of\ small\ circle\ = \pi {r_2}^2 = \pi \left(\frac{10}{2\pi}\right)^2 = \frac{25}{\pi}\ u^2$$

Here, while total of areas of two aluminum bases formed by large circles that are required for a cylinder with a volume of $\mathcal{V}_1$ is $\frac{200}{\pi}\ u^2$, the total of area of four aluminum bases formed by small circles that are required for a cylinder with a volume of $\mathcal{V}_2$ is $\frac{100}{\pi}\ u^2$. It is understood from the calculations that there is an advantage in using small circles, and once this result is supported with different numerical examples, it can be observed that the circle area per volume increases for $\mathcal{V}_1$ as long as the length of the long edge (a) is kept fixed and length of short edge (b) is shortened.

Technology plays a key role in producing mathematical models for the solutions of real-life problems and understanding the relationships (Yerushalmy, 1997)[2]. As seen in Figure 2, Dynamic Geometry software can be used as an effective tool in investigating the problem solution. However, the study was based on paper and pencil since the participating preservice teachers had not received enough training with the software.

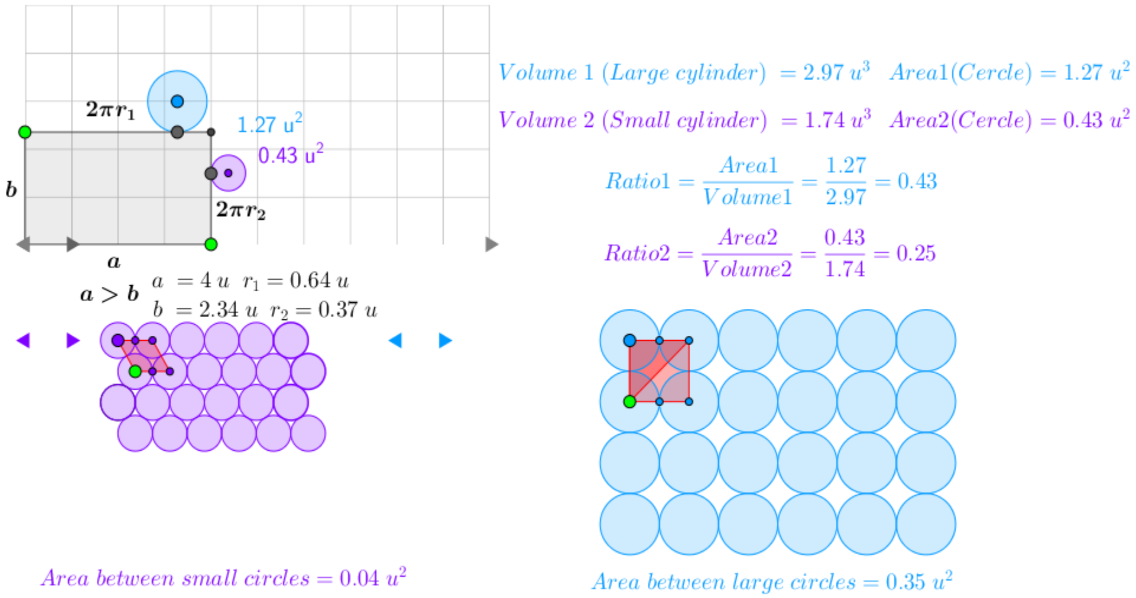


Figure 2. Investigating the problem solution on Dynamic Geometry (GeoGebra)

[2] Yerushalmy, M. (1997). Mathematizing verbal descriptions of situations: A language to support modeling. *Cognition and Instruction, 15*(2), 207-264. doi:10.1207/s1532690xci1502_3