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## Identification of Graph Thinking in Solving Mathematical Problems Naturally

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This study attempts to describe characteristics of graph thinking in solving a mathematical problem. Three students at the 10th grade of senior-high schools were involved as the subject. The data was collected from the result of an optimization problem task (OPT), video recording, interviews, and field notes. The results showed two major characteristics of graph thinking were found in solving the problem. First, students used the concept of graph theory to create a problem modelling. They were able to represent the information given in the problem in the form of graph. Second, students also used the concept of graph theory to create a problem modelling and search algorithm. The problem modelling was created as the students interpreted the problem by making connection between the objects in the form of an adjacency matrix and connectivity. In devising a plan, the students referred to the problem modelling to develop search algorithms. However, the algorithms were not entirely efficient. Some of them required the students to initially describe all answer possibilities. The algorithms constructed by the students referred to sequential and conditional algorithms. This study argues that graph-thinking skill can be developed through a learning process which involves students in the solving of open-ended problem to stimulate ideas of problem solving. By developing graph thinking ability, students will be able to analyse and reason information, express mathematical ideas,

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and have flexibility in solving a problem. These skills are urgently needed in the 21st century where rapid and continuous changes occur.

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## **Introduction**

Thinking is an important activity in human life. In everyday life, people cannot be separated from thinking activities. Through thinking, people are able to gain meanings or understanding about everything they face in life. Thinking refers to information processing which involves mental activities such as judgement, abstraction, reasoning, imagination, and problem solving (Solso et al., 2005).

Mathematics learning at school is acknowledged as a means of clear, critical, creative, systematic, and logical thinking activities. Once mathematics is considered as a means of thinking, it no longer functions as a product that must be given to students. However, the product of mathematical thinking comes as a result of thinking process in learning mathematics. Students will construct concepts or problem solving in their mind in diverse ways. When students encounter mathematical problems, they will be likely to carry out mathematical thinking activities in providing ideas or solutions.

Mathematical thinking refers to a dynamic process which allows individuals to improve the complexity of ideas and develop their knowledge of mathematics (Mason et al., 2011). It is because the process provides opportunities to increase the complexity of ideas from time to time. Meanwhile, Tasdan et al. (2015) assert that mathematical thinking refers to a way of thinking which is related to a mathematical process (doing math) in solving both simple and complex mathematical tasks. Mathematical thinking also demonstrates the importance of understanding about mathematical concepts, the ability to solve mathematical problems, and how to learn on their own and develop the skills needed in independent learning (Katagiri, 2004; Schoenfeld, 2016). Taking these definitions into account, mathematical thinking skill can be defined as a process of thinking which involves the ability to collect and analyse information, make generalizations to develop understanding, and acquire new knowledge.

Mason & Johnson-Wilder (2004) argue that when a mathematics expert deals with a mathematical problem, he will take several stages and actions known as mathematical thinking, and it includes exemplifying, specializing, completing, deleting, corrected, comparing, sorting, organizing, changing, varying, reversing, altering, generalizing, conjecturing, explaining, justifying, verifying, convincing and refuting. Stacey (2014) states that the ability to apply mathematical thinking in solving a problem is the main goal of mathematics education. In this case, the ability to think mathematically will enhance science, technology, economic life as well as its development. With regard to the challenges in workplaces, job seekers must have teamwork skills, the ability to use technology, and more importantly problem-solving skills (English & Gainsburg, 2015; Weigand, 2010). Thus, in its application in learning mathematics, it is essential for students to apply the process of mathematical thinking in solving mathematical problems.

Problem-solving skill is indeed the most essential cognitive activity, and it becomes an integral part of mathematics (Elia et al., 2009; NCTM, 2000). Besides, problem-solving skill is regarded as a cognitive process of seeking solutions to specific problems (Düşek & Ayhan, 2014; Wang & Chiew, 2010). A problem refers to an individual's efforts to solve it in which the individual does not know the solution and tries to solve it. In fact, a problem will not stay forever. A problem we encounter today will not necessarily continue in the future. As stated by Kirschner et al. (2006), a problem can be tackled if we find the solution. Problem can also be interpreted

as a condition in which the objectives as well as the stages to achieve the objectives arise indirectly. Mathematical questions are considered as a problem if they cannot be directly solved. Problem solving not only involves a process of imitating how to solve a problem which students do not know, but students also need to devote extra works such as changing the problem into a familiar one to solve it, dividing the problem into parts, or re-formulating a non-routine problem into the familiar one. Thus, mathematics basically can be seen as a process of problem solving and understanding mathematic requires teaching it through problem solving (Van De Walle, 2004). The most common model used to solve word problems is the one developed by Pólya's presented in a book entitled "How to Solve It" (1945). The model entails four steps: understanding the problem, making plans, implementing the plans, reviewing the results.

Problem solving holds an essential role in mathematical thinking, and some literature argues that graph theory, a branch of mathematics, is widely used to solve problems in everyday life (Dafik et al., 2020; Medová et al., 2019; Nabiyeu et al., 2016). Graph theory is able to represent a complex problem into the easier one to understand and to solve. The implementation of graph theory can be easily found in diverse aspects of human life related to both natural and social sciences. Many real-life problems can be modelled using graph theory. However, graph theory has not been introduced in mathematics at elementary school or high school level in Indonesia. In fact, learning graph theory is able to develop students' analytical skill, critical, and creative skills which are essential for surviving in the 21st century (Burrus et al., 2013; Darling-Hammond et al., 2020; Živković, 2016). These skills obviously contribute to the development of 4C skills which consist of critical thinking, creativity, collaboration, and communication (Dafik et al., 2020).

Graph theory is a branch of discrete mathematics, which has been developing rapidly because of its ability to be applied to various branches of mathematics to solve problems in everyday life. This theory was firstly developed by Leonhard Euler (1736), a Swiss mathematician and physicist, who eventually found the answer to a prominent problem at that time, Konigsberg bridge. Hutchinson et al. (1991) states that a graph simply consists of points known as vertexes and the line connecting these points known as sides. Graph  $G$  is mathematically defined as a set pair  $(V, E)$  in which  $V$  refers to a non-empty set of all points  $(V = \{v_1, v_2, \dots, v_n\})$ , and  $E$  refers to the set of sides connecting a pair of points  $(E = \{e_1, e_2, \dots, e_n\})$ . This notion is directly related to the concept of mathematical modelling which is emphasized in mathematics learning program. Mathematical modelling refers to a process of representing and explaining a problem, phenomenon, or an event in the real word into a mathematical statement so that a more proper understanding of the problem is achieved (Abassian et al., 2020; Doğan et al., 2019).

Basically, graph theory has been introduced to students at elementary or high school level when they learn about a food chain, match scheduling, and organization structure. Similarly, social media like Facebook involves the application of graph theory in everyday life. However, because the theory is not taught in formal education, students do not realize the fact that the theory is closely related to their daily activities. Many problems in everyday life can be modelled into the simpler ones through graph theory.

Graph theory draws attention due to its intriguing ideas and application in various fields. In learning, mathematics is still considered a difficult and less interesting subject. In this regard, graph theory is able to stimulate students' interest in mathematics, help them to use their creative skills and imagination in the process of mathematical investigation, and develop their

ability in applying basic mathematical concepts independently (Niman, 1975). Many mathematical problems in everyday life can be simply solved by using visual representation of points and sides. Thus, mathematical thinking by using the concepts of graph theory holds a critical role in solving mathematical problems.

Mathematical thinking becomes an interesting focus in the studies on mathematics education (Bobis et al., 2005; Cai et al., 2005; Katagiri, 2004; Pérez, 2018; Schoenfeld, 2017; Sung et al., 2017; Tanişli, 2011; van Oers, 2010; E. A. and cooper Warren, 2012). Some researchers designed classroom learning which encourages students to use mathematical thinking (Bobis et al., 2005; Katagiri, 2004; Schoenfeld, 2017; van Oers, 2010). The others conducted studies related to computational thinking (Pérez, 2018; Sung et al., 2017). The results indicate that introducing or integrating computational thinking to mathematics learning is able to improve students' learning outcomes in other subjects. Meanwhile, Cai et al. (2005) and Windsor (2010) carried out a study related to algebraic thinking in students at elementary level. The results reveal that giving mathematical problems is able to prompt students' meaningful experience in developing their algebraic thinking. In addition, Warren et al. (2006) and Tanişli (2011) conducted studies which aimed at identifying functional thinking in elementary students, and the results show that the students are able to apply functional thinking in solving mathematical problems.

In addition to mathematical thinking, the issue of problem solving has been increasingly investigated, as well. A few studies are concerned with mathematical-problem solving in word problems (Csíkos et al., 2012; Goulet-Lyle et al., 2020; Pongsakdi et al., 2020). The study conducted by Goulet-Lyle et al. (2020) concludes that solving word problems through sequential method is unable to develop students' ability in problem solving. Another study shows that the performance of solving word problems on easy and difficult questions is strongly interrelated to students' text comprehension and arithmetic skills (Pongsakdi et al., 2020). In easy questions, students showing lack of text comprehension, but having good numeracy tend to show better performance than those showing good comprehension but having low numeracy. Meanwhile, the study conducted by Csíkos et al. (2012) indicates the importance of visual representations in mathematics modelling in solving word problems.

The application of graph theory in various fields in daily activities has been widely investigated, and the focuses include computer science (Cvetković & Simić, 2011; Singh & Vandana, 2014), health (Mears & Pollard, 2016; Vecchio et al., 2017), transportation (Lanjewar et al., 2015; Nelson et al., 2019), geology model (Phillips et al., 2015), banking (Lin et al., 2011), architecture design (Poyias & Tuosto, 2015; Zboinska, 2015), regional planning (Foltête et al., 2014), biology (Gao et al., 2018; Garroway et al., 2008), and the electrical system (Benchouia et al., 2014; Šarga et al., 2012). In addition, several studies investigated the use of graph theory in learning process in class (Asghari et al., 2012; Medová et al., 2019; Nabiyev et al., 2016; Uyangör, 2019). Asghari et al. (2012) in their study show that graph theory allows students to think creatively, construct a collection of ideas, and use graph representations in solving problems. Uyangör (2019) in his study pinpoints that integrating graph theory in the process of learning mathematics can improve students' achievement in the stages of mathematical thinking. Besides, Nabiyev et al. (2016) also underpinned that the use of both artificial intelligence technique and graph theory is able to encourage students to solve complex word problems easily in the problem-solving process. Meanwhile, the study conducted by Medová et al. (2019) identifies errors related to graph algorithms and compare them based on the types of error. The results indicate that from three problems given, most students encounter problems in *Chinese Postman Problem* rather than those in *The Shortest Path Problem* and *The Minimum*

### *Spanning Tree Problem.*

The facts from those studies show that there is still no study investigating mathematical thinking in solving an optimization problem by involving graph theory. Therefore, the concern of this study is the "Identification of Graph Thinking in Solving Mathematical Problems Naturally". In this case, graph thinking refers to a way of thinking in solving an optimization problem that has never occurred previously through the basic concepts of graph theory naturally. In addition, natural problem solving is a way of thinking, analysing, and reasoning carried out by students to understand and solve a non-routine and open-ended problem by using their experience and abilities (Csapó & Funke, 2017). This study attempts to address the question of “*what are the characteristics of students’ graph thinking in solving mathematical problems naturally?*”. This study is an initial study of a graph thinking process in solving mathematical problems. The purpose of this study is to describe the characteristics of graph thinking in solving mathematical problems.

## **Methodology**

### ***Research Design***

This study used a qualitative model with a descriptive exploratory approach. Several characteristics of this model, based on Creswell, John W. & Creswell (2018) are natural setting, the researcher as a key instrument, multiple data sources, inductive data analysis, emergent designs, and holistic account. The characteristics of students’ graph thinking were identified based on the Optimization Problem Task (OPT) results of those who met the criteria in giving solution and used basic concept of graph theory. The data was accompanied by interview results and video transcription. During the interview, the students were given the opportunity to express their idea freely so that the characteristics of their graph thinking can be easily identified. The interview was carried out to uncover the problem-solving strategies used by the students, as well. The results of the analysis were used to describe the characteristics of graph thinking in the process of solving the optimization problem. To draw conclusions, the interview results were synchronized with the students’ answers. The researchers also confirmed the validity of the data by ensuring the accuracy and completeness of the data collected as well as validating the coding and recording process from different characteristics through expert discussions.

### ***Participants***

The subjects of in this study were tenth-grade students at senior high schools. They were derived from three different high schools in Cirebon, Indonesia. One class was selected from each school based on the recommendations given by the mathematics teachers. The distribution of the subjects is presented as follows.

**Table 1.** The distribution of the subjects from three different schools

No	School	Number of Students
1	Senior High School 1	27
2	Senior High School 2	28
3	Senior High School 3	30
Total		85

The subjects of this study were chosen based on the following criteria. (1) The subjects were at



the tenth-grade of senior-high schools. Students at this grade are considered to be at the stage of formal operational thinking and able to reason logically as well as work effectively and systematically. (2) The subjects were able to communicate ideas effectively and communicatively. (3) The subjects were willing to be involved in the data collection process so that the data was more accurate. (4) The subjects had the ability to solve the problem given. (5) The subjects were able to depict the problem in the form of graph representation. With regard to these criteria, three students were chosen out of 85. Two of the three students were considered to represent the characteristics of graph thinking due to their ability to depict the and develop algorithm problem in the form of graph. The problem-solving ability of the subjects were measured based on the answer and the process of solving the problem. Therefore, the analysis results gave a vivid description of the students' graph thinking characteristics in solving the problem. Three students from three different schools were eventually chosen. In order to distinguish those three students in the result and discussion section, each student was given the initial S (S1= Subject 1; S2= Subject 2; and S3= Subject 3).

**Instrument**

the researchers collected the data through audio recording, documentation, observation, and interviews with the students and they interpreted the findings of this study. Besides, Optimization Problem Task (OPT) was employed as a supporting instrument. The task given was concerned with graph colouring, to which the students have never been exposed previously. The supporting instrument is depicted in the following figure.

There are 7 types of chemicals that need to be stored in a warehouse. Several pairs of these substances cannot be stored in the same room because the gas mixture is easy to explode (see table chemical correlation). These substances need to be stored in separate rooms equipped with different vents and exhausts. The more space it takes, means more costs that must be incurred. What is the minimum amount of space required to store all chemicals safely?

Chemicals	Cannot be stored with chemicals
A	B, D
B	A, D, E, F, G
C	E, G
D	A, F, B
E	B, C, G
F	B, D
G	C, E, B

**Figure 1.** Optimization Problem Instrument

The instrument employed in this study was derived from the one developed by Bacak (2004) which was initially used to examine the concept and theorem of vertex coloring. As this study involved senior-high school students, the instrument was adjusted by considering the difficulty level of the problem based on the standard of mathematics curriculum in Indonesia. The instrument used in this study was also validated by experts.

### **Data Collection**

The OPT was used as a tool to obtain the data regarding the students' graph thinking characteristics on optimization problems. This study was undertaken in several stages as follows: before the students did the OPT, the researchers firstly explained the general objectives of this study and asked for their permission for being the subjects of this study without explaining that they would be re-selected for the next stage based on their answers. The subject worked on and completed the OPT independently in 30 minutes. The subjects' answer sheets were corrected and analysed to identify the characteristics of their graph thinking in solving the optimization problem through the basic concept of graph theory. Giving such "non-routine" problem was intended to explore the natural way of graphs thinking in solving the optimization problem. The results of the subjects' works were further categorized based on the method of completion which represented the characteristics of graph thinking.

Task-based interviews were then conducted in order to explore the characteristics of graph thinking. The interview results aimed at explaining the mental processes taken by the subjects. In conducting the interviews, the researchers directly asked for the confirmation from the subjects regarding the steps taken to complete the OPT. The interviews were recorded in both in the form of audio and video files.

### **Data Analysis**

Before carrying out comprehensive analyses, the data was triangulated by comparing the data collected through the OPT and interviews in order to confirm the data validity (Creswell, John W. & Creswell, 2018). The data analyses were carried out in two stages. First, the students' answer sheets on the OPT were analysed by looking at the methods used to solve the problem and dividing the subjects in terms of the problem modelling and the use of problem-solving algorithm involving the concepts of graph theory. Second, the interview results were analysed by transcribing the recordings. The transcriptions enabled the researchers to comprehensively identify the characteristics of graph thinking in solving the problem. In short, the results of the subjects' answers and the interviews were analysed simultaneously to provide comprehensive explanation.

### **Results**

This section presents the results of the student works in completing the OPT and the interview results. The students who have never been exposed to the optimization problem were assigned to do the optimization problem which was generally solved through the concepts of graph theory. The results of this study also elaborated the characteristics of graph thinking in solving the OPT shown by three students divided into two groups. The first group consisted of two students who were naturally able to create problem modelling in the form of a graph. Each student was given a code: S1 (the student from the first school) and S2 (the student from the second school). The second group consisted of the student who was naturally able to create problem modelling and organize a structured algorithm search involving the concepts of graph theory. This student was coded as S3 (the student from the third school). The elaboration was dominated by researchers' interpretation on the students' written answers. In addition, the interview results provided supporting findings regarding the characteristics of graph thinking in solving the OPT.

**Modelling the problem in graph form naturally by S1 and S12**

In demonstrating the steps in solving the OPT, S1 initially transferring the information into a new table as displayed in Figure 2. In this table, S1 presented the connection of each chemical substance by using signs [•], [v], and [x]. The graph thinking applied by S1 in solving the problem was modelling the problem into a table of connection between the points and sides. The interview result with S1 is shown in the following transcription.

- Interviewer : How did you complete the task?  
 S1 : Hmm... I followed the instructions. I tried to make a new table to enable me to solve the problem. [showing the new table. See Figure 2].  
 Interviewer : So, what's wrong with the table provided in the question? Is the information hard to understand and determine the minimum amount of space?  
 S1 : Yes, sir. I find it difficult to understand if I don't change it into a new table.  
 Interviewer : What do the signs [•], [v], and [x] mean in the new table you have made?  
 S1 : Oh yeah, the sign [•] shows a pair of same substances. The sign [v] indicates that the pair can be stored in the same room, while the sign [x] means that the pair cannot be stored in the same room.  
 Interviewer : How does your table work to solve the problem?  
 S1 : Hmm... I determine the answers based on the signs [v] and [x] in each column. The column which has the highest similarities in terms of the signs [v] or [x] can be paired.  
 Interviewer : Can you explain how it works? How is A finally paired with E and F?  
 S1 : First, I look at which column the mark [v] is assigned to in the row A. Because the mark [v] is assigned to C, E, F, and G, I determine which column shares similarities with A in terms of the signs [v] and [x]. Then, because C only have one similarity, I decided to separate C from A. The same way is used to separate G from A. As a result, only E and F are possible to be paired with A.  
 Interviewer : Why are E and G treated differently? In fact, E and G share similarities in terms of the characteristics or marks with A.  
 S1 : Hmm... yes, you're right sir. [unable to provide further elaboration]  
 Interviewer : Okay. Do you use the same way to determine the other pairs of chemical substances?  
 S1 : Yes, sir. I do the same thing.

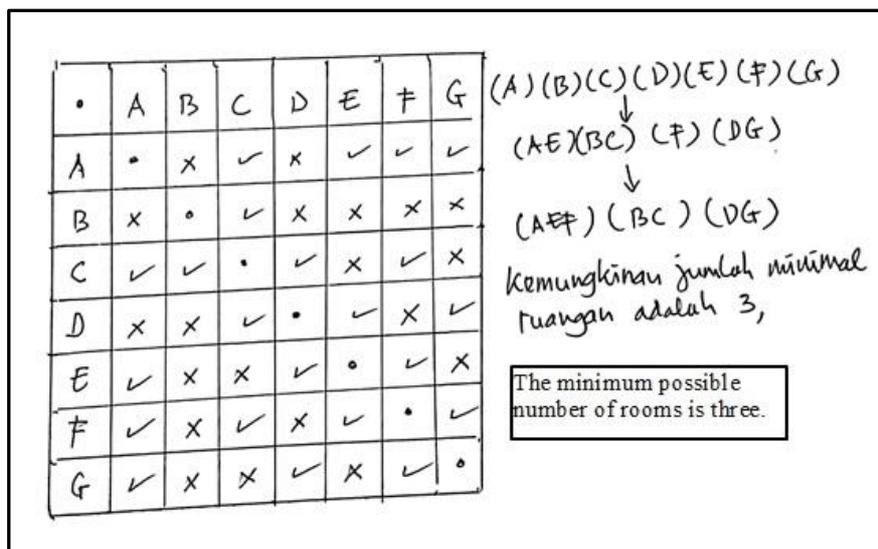
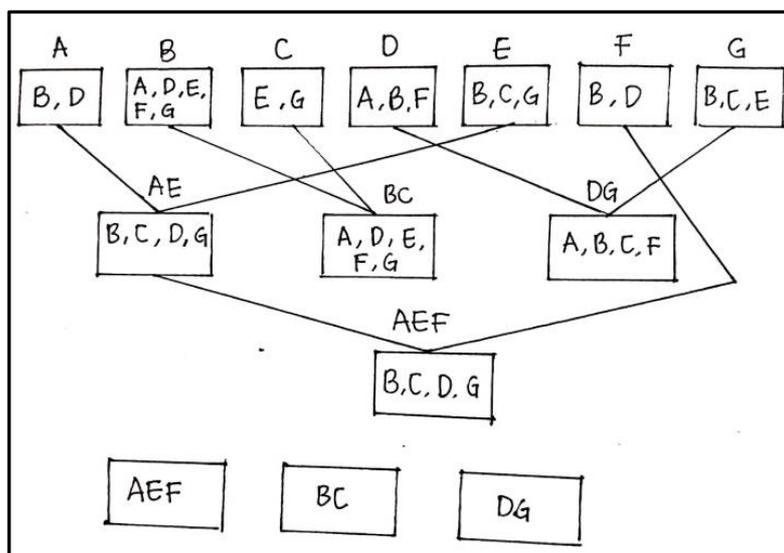


Figure 2. Problem modelling created by S1 in solving the OPT

Figure 2 depicts that in solving the optimization problem, S1 presents the information given in the question in more simple and understandable ways. Based on the explanation given by S1, he models the problem in a table connecting the characteristics of each chemical substance by giving signs [•], [v], and [x]. The table not only make the problem more understandable, but it is able to determine the minimum number of rooms needed to store the substances by looking at similar signs in each column. Despite the good model, the rules of algorithm search used by S1 are considered less logical and structured. The interview results also revealed that S1 finds it difficult to explain the criteria in determining which substances can be stored together according to the model he made.

Furthermore, S2 creates a model which connects points and sides. The model displays the relationships between the characteristics of each substance (see Figure 3). This implies that the connection is not only concerned with two objects connected but also the characteristics of each object. The interview result with S2 is presented as follows.

- Interviewer : How do you solve the problem?  
S2 : I try to connect each substance and the one which cannot be paired [showing the visual representation as presented in Figure 3]  
Interviewer : What do you mean by writing this A [B, D] ? [showing one part of the visual representation]  
S2 : Oh yeah, this means that the substance A cannot be stored together with B and D.  
Interviewer : Can you tell me how you made this connection?  
S2 : Yes, I firstly pair 2 substances. For instance, I make a pair A [B, D], so B [A, D, E, F, G] and D [A, B, F] will have no possibility to be paired with A [B, D]  
Interviewer : Hmm... why don't you pair C [E, G] with A [B, D]?  
S2 : Oh ya, I do not take C [E, G] because there is no similar substance which cannot be stored together. Consequently, I choose E [B, C, G] to be paired with A[B, D].  
Interviewer : Hmm, do you eliminate F [B, D] and G [B, C, E] to be paired with A [B, D]?  
S2 : Yes sir. At the initial stage, I only try to pair two substances, and I choose E[B, C, G].  
Interviewer : Do you have specific criteria to choose between E [B, C, G], F [B, D] and G [B, C, E]?  
S2 : No sir, I choose the pair based on the alphabetical order.  
Interviewer : Then what do you mean by writing A | E [B, C, D, G] in the second line?  
S2 : Oh ya, it means that A is paired with E. Meanwhile, B, C, D, and G cannot be paired with A and E.  
Interviewer : Do you use the same way to determine the other pairs of substances?  
S2 : Yes sir. I use the same way.  
Interviewer : In the third line why do you pair A | E [B, C, D, G] with F [B, D]?  
S2 : I pair F[B, D] with A | E [B, C, D, G] because they have the same intersection in terms of the substances which cannot be paired. As a result, fewer rooms can be obtained.



**Figure 3.** Problem Modelling creating by S2 in solving the OPT

With regard to the Figure 3 above, S2 models the problem through a graph connecting the characteristics of each substance. Each side combines the substances which have similarities or intersections – having no possibility to be paired with a certain substance. For instance, the combination A [B | D] means that A cannot be paired with B and D. At this point, a connection is given to E [B, C, G] which means that A can be paired with E so that the points are combined into A | E [B, C, D, G]. The same way is carried out until all substances are connected, and the optimum solution is achieved. In explaining the procedure for solving the problem, S2 find it difficult to explain the logical criteria for choosing between [B, C, G], F [B, D] and G [B, C, E] to be paired with A [B, D].

### **Modelling and designing algorithms in graph form naturally by S3**

S3 remodels the information given in the optimization task in the form of graph connecting points and lines as displayed in Figure 4. The points represent chemical substances, while the lines indicate that two substances cannot be paired. The connection graph enables S3 to understand the problem and finding solutions. Through the graph, S3 follows the problem-solving steps in a well-structured way, and the interview result explains how the steps work.

Interviewer : How do you solve the problem?

S3 : I firstly depict the relationships between each substance in order to make it easier to complete the task [showing the table]

Interviewer : Can you explain about the picture?

S3 : I make seven points according to the number of chemical substances presented in the problem. Then, I connect the substances which cannot be paired .

Interviewer : Do you use any sign such as a triangle, rectangle, or circle in presenting the substances?

S3 : Yes sir, I purposely use triangles, rectangles, and circles to differentiate each substance. The substances which are directly related are given different signs. It means that the substances cannot be paired [showing Figure 4]

Interviewer : Can you tell me how to determine the signs?

S3 : Firstly I give a rectangular sign to A. Then, I give the same sign to the substances which are not directly related to A (possibly C, E, F, and G). Then, F and G are

- given the same sign as A. This means the same sign cannot be given to C and E as they are related directly to F and G
- Interviewer : Is it related to the colors you give in the model?
- S3 : That's right sir. I give the same color (yellow) to the pairs having the same sign (AF, AG, and GF).
- Interviewer : Then, what do you do next?
- S3 : I begin the next step with one of the substances directly related to A, which are B and D. I take B and give it a circle sign. Just like the previous step, I determine another substance which can be given the same sign as B, which is C [showing the pairs colored in red]
- Interviewer : How about the triangle sign?
- S3 : I give a triangle sign to the unpaired substances, which are D and E. Since D and E are not related, they can be given the same sign [showing the pairs colored in purple]
- Interviewer : If D and E are directly connected, will you assign another sign?
- S3 : Oh of course, sir. If they are directly connected, I will assign another sign which represents the fourth room for storing these substances.

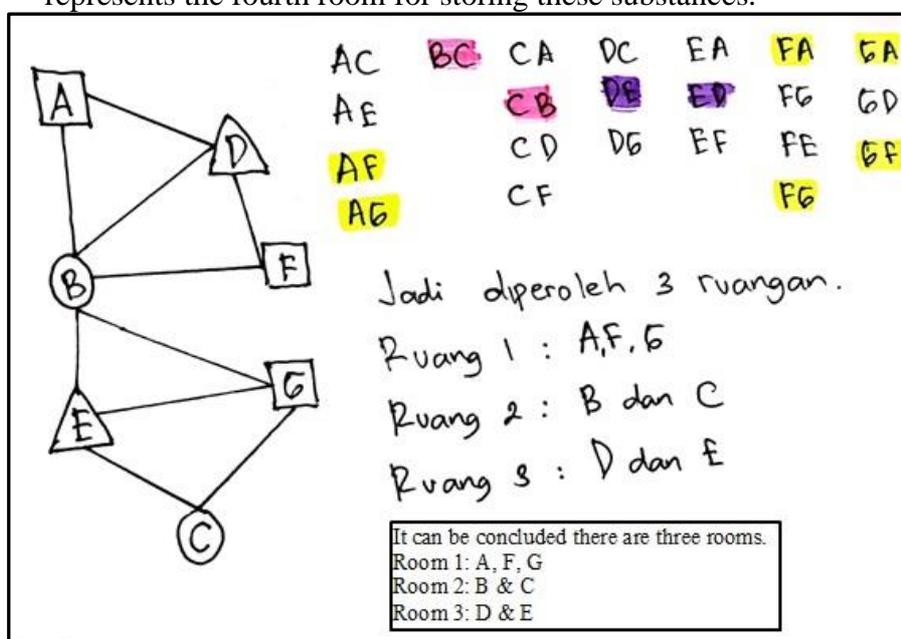


Figure 4. S3's work in completing the OPT

Figure 4 shows that S3 models the problem in the form of a graph connecting points and sides. The points represent chemical substances, and the sides indicate that two substances connected cannot be stored together. The graph not only makes it easier to understand the problem, but it is also used to follow the steps in finding the minimum number of rooms to store the substances. S3 gave different symbols to the substances which are directly connected. This step is carried out in a structured way so that all points are assigned to a certain sign. The same sign indicates that the substances can be stored together in the same room. In addition to using the graph, S3 also gives different colours to determine the minimum number of rooms (see Figure 4). A pair of substances which can be put together in the same room is given the same colours.

## Discussion

The results indicate the way the students think in solving the optimization problem related to the storage of hazardous chemical substances. Even though they have not received graph theory material at school as it is not included in the current mathematics curriculum, the application of graph theory cannot be separated from their daily activities. In addition, the results from the subjects' works show that the students have involved the concept of graph theory in identifying the problem and finding solutions to the optimization problem. This corresponds to the results of several other studies (e.g., Căprioară, 2015; Charlesworth & Leali, 2012; Düşek & Ayhan, 2014; NCTM, 2000) that each individual has a certain way of thinking or approach to solve problems based on their previous experience. One of the initial steps in solving the optimization problem is to firstly understand the problem given. The analysis of the subjects' works indicates that they modelled the problem in the form of graph as carried out by S1, S2, and S3. This shows that modelling the problem in graph helps the students understand the problem and creating plans to solve the problem. In this case, S2 and S3 modelled the problem in the form of a graph connecting points and sides despite showing different connectedness in terms of the sides. Meanwhile, S1 modelled the problem in the form of a table marked [v] and [x]. The marks indicate the connectedness of two chemical substances. If we pay close attention, the problem modelling made by S1 seems to represent another form of graph adjacency matrix in which the matrix component contains number 0 and 1 (Jehu Peters & Don Metz, 2015; Voloshin, 2009). Furthermore, the models made by the three subjects are the representations of the problem in graph form. This corresponds to the research results (Asghari et al., 2012; Medová et al., 2019; Uyangör, 2019) showing that modelling in the form of a graph is able to describe mathematical relationships through real-world problems in order to be more easily understood if they are reflected in graph representations. Moreover, the models made by the subjects are able to help them find solutions to solve the problem and stimulate them to solve more sophisticated problems.

In addition to the problem modelling in graph form, other findings based on the subjects' answers indicated the use of structured algorithms in solving optimization problems. Only S3 demonstrated the use of an algorithm based on coherent modelling. The problem-solving algorithm was employed by S3 to give different symbols for each pair of neighbouring points in the model. This step was carried out repeatedly until each point had a symbol, and the symbol was different from one another. The use of algorithm in the problem-solving process might be a common thing if the students have already known the algorithmic steps and solved the same problems previously. However, the findings of this study show that in solving the optimization problem, S3 used an algorithm problem-solving based on the problem modelling. This shows an intriguing finding since S3 had never encountered similar problems previously or been exposed to materials regarding graph theory at school. In other words, S3 was naturally able to identify the problem in the form of graph and organize structured algorithm solving. One of the problem-solving process standard (NCTM, 2000) is that a good problem solver tends to be naturally able to analyse situations thoroughly in mathematical relationships and solve the problems based on the situation they engage in. This corresponds to the results of several studies (Elia et al., 2009; Medová et al., 2019) which show that problem-solving steps taken by S3 involved a decision-making process regarding the tools he used, the way he used them, as well as flexible thinking. S1 and S2 basically organized problem-solving algorithm in less structured and logical ways. This might be caused by the fact that they are not familiar with the optimization problem as stated by Rao (2020) and Medová et al. (2019).

Problem-solving skill becomes one of the abilities which must be acquired during the era of industrial revolution 4.0 and even in the upcoming era in which information technology and the

internet are used in every activity (Puncreobutr, 2016; Wollschlaeger et al., 2017). This situation requires students to have the ability to adapt and prepare their competencies, especially the ability to think about how technology solves a problem. In this case, students must have good algorithm thinking skill like well-structured and logical algorithm on a computer. Therefore, graph thinking can be seen as an ability to deal with problems, and it is urgently required in this industrial revolution era 4.0.

With regard to the way graph thinking is related to problem-solving as well as its flexibility nature to be applied across disciplines, mathematics can be considered as a relevant field as a means of developing students' skills related to this issue. This is due to the fact that mathematics requires students to think logically and find a problem-solving (Das et al., 2020; Pongsakdi et al., 2020; Sutarto, Toto Nusantara, Subanji, 2016). In terms of its relationship with thinking skills in mathematics learning and related learning theories, graph thinking is closely related to creative thinking, problem solving, abstract thinking, iteration, patterns, synthesis, and metacognition (Burrus et al., 2013; Darling-Hammond et al., 2020; Živkovič, 2016). Thus, it is obvious that mathematics and its learning can be applied to develop graph thinking in solving problems.

## **Conclusions**

Based on the discussion, it can be concluded that in solving an optimization problem the students were required to be able to identify the problem carefully. To solve the problem, the students can simplify the problem by creating problem modelling of the information. The problem modelling created by the students was identified as a representation of graph including connectivity, table, and adjacency matrix. With regard to the problem modelling created, the students constructed different algorithms, specifically in the form of sequential and conditional ones. The problem modelling in the form of connectivity was able to facilitate systematic and efficient problem-solving stages. In solving problem, two major characteristics of graph thinking were identified. They were in the form of problem modelling and algorithms. The graph-thinking characteristics of each student was determined by their previous knowledge and experience.

This study is still limited as there were only three students chosen based on the way the created problem modelling, the way they used algorithms, and the way they gave final solutions. Therefore, further studies are needed to obtain general overview of graph-thinking characteristics in solving a problem. One of the important skills needed by students to solve a complex problem is literacy in understanding the problem. Consequently, further studies also need to embrace literacy skill in creating problem modelling. In addition, teachers are highly recommended to involve students in the process of solving open-ended problems in order to stimulate problem-solving ideas. Teachers also need to apply learning which integrates several concepts of graph theory in teaching mathematics. Through this integration, it is expected that graph theory is able to develop students' analytical, critical, and creative skills as a cornerstone to successfully survive in the 21<sup>st</sup> century.

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